EARTHQUAKE PRESSURES ON FLUID CONTAINERS

by

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EIGHTH TECHNICAL REPORT

under

OFFICE OF NAVAL RESEARCH

Contract N6 onr-244

Task Order 25

PROJECT DESIGNATION NR-081-095

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA

August 1954

TABLE OF CONTENTS

1. Introduction. The dynamic fluid pressures developed during an earthquake are of importance in the design of structures such as dams, tanks and caissons. The first solution of such a problem was that by Westergaard (1933) who determined the pressures on a rectangular, vertical dam when it was subjected to horizontal acceleration. Jacobsen (1949) solved the corresponding problem for a cylindrical tank containing fluid and for a cylindrical pier surrounded by fluid. Werner and Sundquist (1949) extended Jacobsen's work to include a rectangular fluid container, a semicircular trough, a triangular trough and a hemisphere. Graham and Rodriguez (1952) gave a very complete analysis of the impulsive and convective pressures in a rectangular container. Hoskins and Jacobsen (1934) measured impulsive fluid pressures and Jacobsen and Ayre (1951) gave the results of similar measurements. Zangar (1953) presented the pressures on dam faces as measured on an electrical analog.

The foregoing analyses were all carried out in the same fashion, which requires finding a solution of Laplace's equation that satisfies the boundary conditions. With these known solutions as checks on accuracy it is possible to derive solutions by an approximate method which avoids partial differential equations and series and presents solutions for a number of cases in simple closed form. The approximate method appeals to physical intuition and makes it easy to see how the pressures arise. It thus seems to be particularly suitable for engineering applications.

To introduce the method the problem of the rectangular tank is treated in some detail. Applications to other types of containers are treated more concisely. The essence of the method is the estimation of a simple type of flow which is similar to the actual fluid movement and this simple flow is used to determine the pressures. The method is analogous to the Rayleigh-Ritz method used in the theory of elasticity, and it always overestimates the forces. The method is capable of solving a wide variety of problems but if it is required that the solutions be in simple form, which they should to be practically useful, the number of problems that can be handled satisfactorily are limited, just as in the case of the Rayleigh-Ritz method. Acknowledgement is due C. M. Cheng

for carrying out the calculations in this report.

2. Impulsive Pressures. Rectangular Tank. Consider a rectangular container as shown in Figure 1, and at the instant under consideration let the surface of the fluid be horizontal and let the walls of the container have a horizontal acceleration $\dot{u}_0$ in the x-direction. Let it be required to find the pressures on the walls of the container due to the acceleration $\dot{u}_0$. Let the fluid have a depth 'h', a length '2ℓ' and a unit thickness. It is seen that the action of the fluid is similar to that which would be obtained if the horizontal component of fluid velocity, u, were independent of the y coordinate; that is, imagine the fluid to be constrained by thin, massless, vertical membranes free to move in the x-direction, and let the membranes be originally spaced a distance dx apart. When the walls of the container are given an acceleration the membranes will be accelerated with the fluid and the fluid will also be squeezed vertically with respect to the membranes. As shown in Figure 2, since the fluid is restrained between two adjacent membranes, the vertical velocity v is dependent on the horizontal velocity u according to

$$v = (h - y) \frac{du}{dx} \tag{1}$$

This is an equation specifying the constraint on the fluid flow. As the fluid is considered incompressible it follows that the acceleration $\dot{v}$ is proportional to the velocity v and the acceleration $\dot{u}$ is proportional to the velocity u, and the pressure in the fluid between two membranes is given by the standard hydrodynamical equation:

$$\frac{\partial p}{\partial y} = -\rho \dot{v} \tag{2}$$

where ρ is the density of the fluid. The total horizontal force on one membrane is

$$P = \int_0^h p \, dy \tag{3}$$

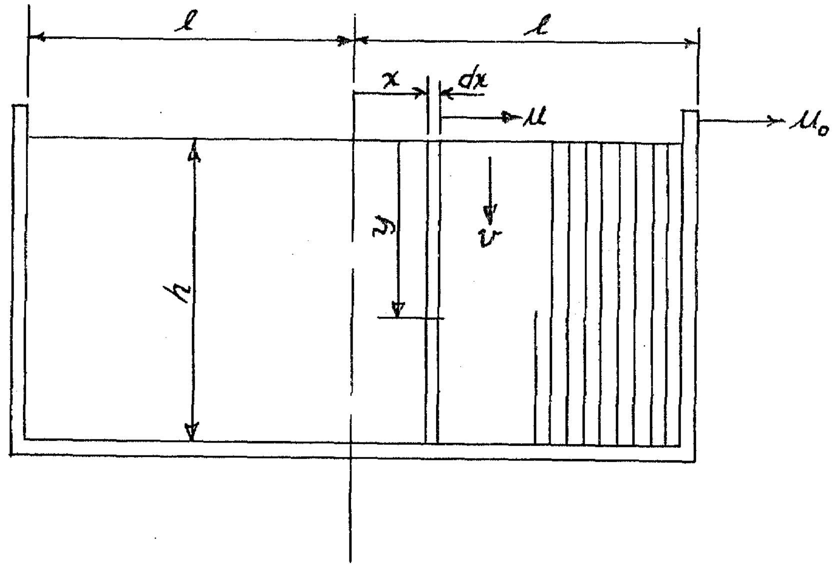

FIGURE 1

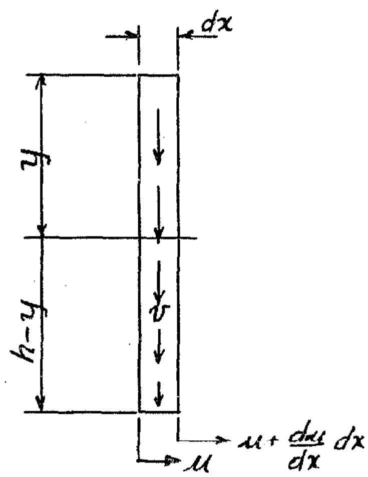

FIGURE 2

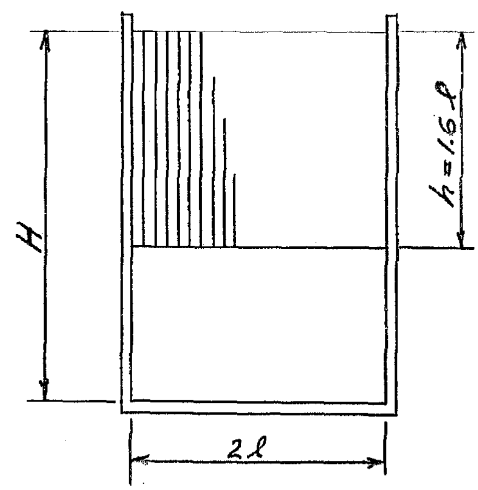

FIGURE 4

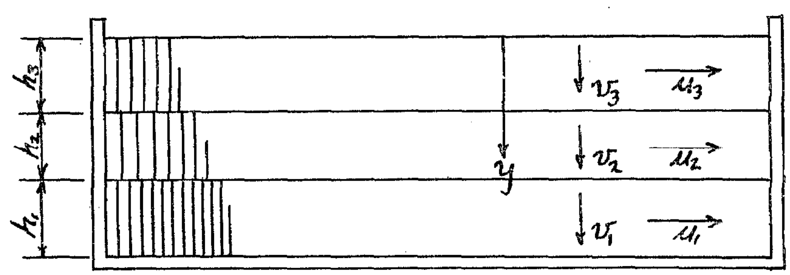

FIGURE 3

These three equations may be written as follows:

$$v = (h-y)\frac{du}{dx}$$

$$p = -\rho\int_0^y (h-y)\frac{du}{dx}\,dy = -\rho h^2\left(\frac{y}{h}-\frac{1}{2}\left(\frac{y}{h}\right)^2\right)\frac{du}{dx}$$

$$P = -\rho h^2\int_0^h \left(\frac{y}{h}-\frac{1}{2}\left(\frac{y}{h}\right)^2\right)\frac{du}{dx}\,dy = -\rho\frac{h^3}{3}\frac{du}{dx} \tag{4}$$

The problem is thus solved when the velocity u is known.

The velocity u will first be determined formally for purposes of illustration and will then be deduced in a less formal but somewhat simpler manner.

The kinetic energy of the fluid is

$$T = \int_{-\ell}^{+\ell}\int_0^h \tfrac{1}{2}\rho(u^2+v^2)\,dx\,dy$$

and the potential energy of the fluid is zero. Hamilton's Principle states that

$$\delta\int_{t_1}^{t_2}(T-V)\,dt \quad = 0$$

which for this problem is

$$\frac{1}{2}\rho\int_{t_1}^{t_2}\int_{-\ell}^{+\ell}\int_0^h \delta\left(u^2+(h-y)^2\left(\frac{du}{dx}\right)^2\right)dy\,dx\,dt$$

or, integrating with respect to y,

$$\frac{1}{2}\rho\int_{t_1}^{t_2}\int_{-\ell}^{+\ell}\delta\left(u^2 h + \frac{h^3}{3}\left(\frac{du}{dx}\right)^2\right)dx\,dt$$

Carrying out the variation, we obtain

$$\int_{t_1}^{t_2}\int_{-\ell}^{+\ell}\left(2u\,\delta u + \frac{2}{3}h^2\frac{du}{dx}\,\delta\left(\frac{du}{dx}\right)\right)dx\,dt$$

Integrating the second term by parts and then equating the coefficient of (δu) to zero gives the differential equation for u :

$$\frac{d^2u}{dx^2} - \frac{3}{h^2} u = 0 \tag{5}$$

If this equation does not give a sufficiently accurate solution an improvement can be achieved by subdividing the fluid into three regions as shown in Figure 3. Instead of equation (1) there will now be the following equations:

$$\begin{aligned} v_1 &= (h_1 + h_2 + h_3 - y) \frac{du_1}{dx} \\ v_2 &= (h_2 + h_3 - y) \frac{du_2}{dx} + h_1 \frac{du_1}{dx} \\ v_3 &= (h_3 - y) \frac{du_3}{dx} + h_2 \frac{du_2}{dx} + h_1 \frac{du_1}{dx} \end{aligned} \tag{6}$$

Applying Hamilton's Principle leads to three simultaneous equations in u_1 , u_2 , u_3 . Thus introducing additional degrees of freedom leads to more complicated mathematics but will improve the accuracy, for it is clear that in the limit as the subdivisions approach 'dy' the accuracy becomes perfect.

Let us now derive equation (5) in a more straightforward fashion. The slice of fluid shown in Figure 2 will be accelerated in the x-direction if the pressures on the two faces differ. The equation of motion is

$$\frac{dP}{dx} dx = -\rho h \, dx \, \ddot{u}$$

Using the value of P from equation (4) gives

$$\frac{d^2\ddot{u}}{dx^2} - \frac{3}{h^2} \ddot{u} = 0 \tag{7}$$

The solution of this equation may be written

$$\dot{u} = C_1 \cosh \sqrt{3}\frac{x}{h} + C_2 \sinh \sqrt{3}\frac{x}{h} \tag{8}$$

The boundary conditions are

$$\dot{u} = \dot{u}_0 \quad at \quad x = \pm l$$

which gives

$$\dot{u} = \dot{u}_0 \frac{\cosh \sqrt{3}\frac{x}{h}}{\cosh \sqrt{3}\frac{l}{h}} \tag{9}$$

From equation (4) we obtain

$$p = -\rho \dot{u}_0 h \sqrt{3} \left(\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right) \frac{\sinh \sqrt{3}\frac{x}{h}}{\cosh \sqrt{3}\frac{l}{h}} \tag{10}$$

$$P = -\rho \dot{u}_0 \frac{h^2}{\sqrt{3}} \frac{\sinh \sqrt{3}\frac{x}{h}}{\cosh \sqrt{3}\frac{l}{h}} \tag{11}$$

The acceleration $\dot{u}_0$ thus produces an increase of pressure on one wall and a decrease of pressure on the other wall of

$$p_w = \rho \dot{u}_0 h \left(\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right) \sqrt{3} \tanh \sqrt{3}\frac{l}{h} \tag{12}$$

and produces a pressure on the bottom of the tank

$$p_b = -\rho \dot{u}_0 h \frac{\sqrt{3}}{2} \frac{\sinh \sqrt{3}\frac{x}{h}}{\cosh \sqrt{3}\frac{l}{h}} \tag{13}$$

The total force acting on one wall is

$$P = \rho \dot{u}_0 \frac{h^2}{\sqrt{3}} \tanh \sqrt{3}\frac{l}{h} \tag{14}$$

and it acts at a distance above the bottom

$$h_0 = \frac{3}{8} h \tag{15}$$

It is seen that the overall effect of the fluid on the wall is the same as if a fraction $2P \div 2\ell h \rho \dot{u}_o$, of the total mass of the fluid were fastened rigidly to the walls of the container at a height 3/8 h above the bottom. Calling this equivalent mass M_o we have

$$M_o = M \frac{\tanh \sqrt{3}\frac{\ell}{h}}{\sqrt{3}\frac{\ell}{h}} \tag{16}$$

where M is the total mass of the fluid.

The total moment exerted on the bottom of the tank is

$$\int_{-\ell}^{+\ell} p_b \, x \, dx = -\rho \dot{u}_o h^2 \ell \left(1 - \frac{\tanh \sqrt{3}\frac{\ell}{h}}{\sqrt{3}\frac{\ell}{h}}\right)$$

Including this we find that to produce the correct total moment on the tank the mass M_o must be at an elevation

$$h_o = \frac{3}{8} h \left(1 + \frac{4}{3}\left(\frac{\sqrt{3}\frac{\ell}{h}}{\tanh \sqrt{3}\frac{\ell}{h}} - 1\right)\right) \tag{15'}$$

As the tank becomes tall and narrow the following correction should be made. Consider the tank shown in Figure 4 which has a rigid horizontal membrane at a distance h below the water surface. The moment exerted on this membrane by the fluid above is given by the preceding equation. The moment exerted on the membrane by the fluid below is $\frac{2}{3} \rho \dot{u}_o \ell^3$. Equating these for $\frac{\ell}{h}$ small we obtain $\frac{h}{\ell} = 1.6$. This means that the preceding equations should be used only for tanks whose proportions are $\frac{h}{\ell} \leq 1.6$, and if the tank is taller it should be treated as shown in Figure 4 where the lower portion of the fluid moves as a rigid body exerting a pressure on the walls of

$$p_w = \rho \dot{u}_o \ell$$

The accuracy of the preceding equation can be checked by comparing with the values computed by Graham and Rodriguez (1952). In place of equation (16) they obtained

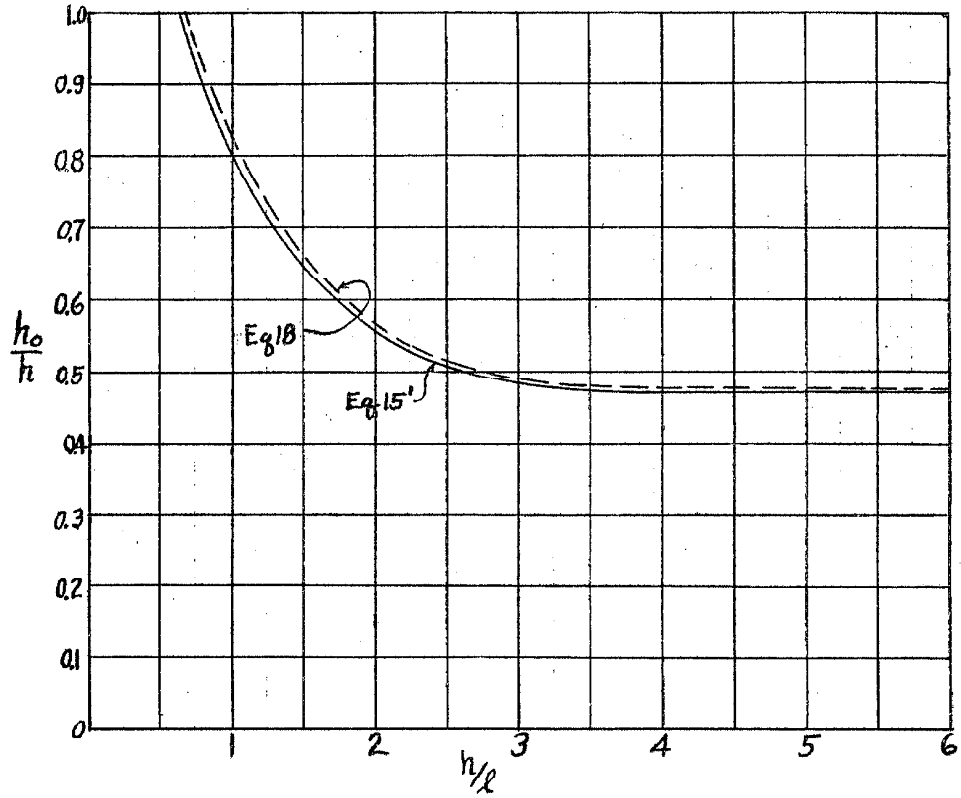

FIGURE 6

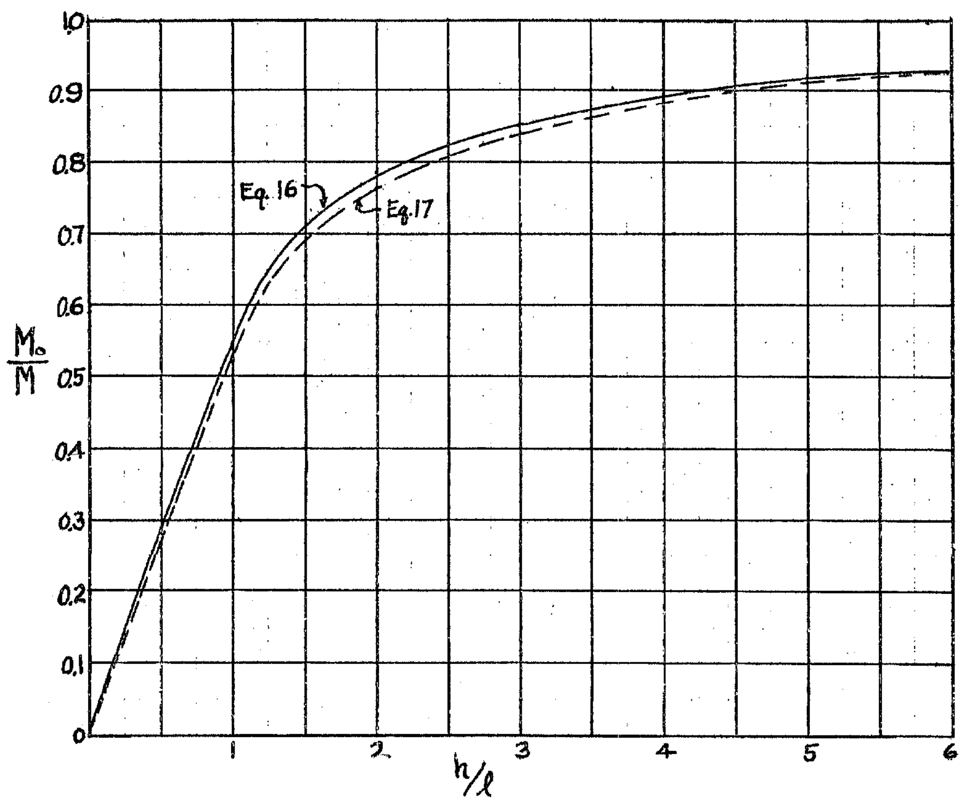

FIGURE 5

$$M_o = M\left[1 - \sum_{n=0}^{\infty} \frac{8 \tanh\{(2n+1)\frac{\pi}{2}\frac{h}{\ell}\}}{\pi^3 (2n+1)^3 \frac{h}{2\ell}}\right] \tag{17}$$

Equations (16) and (17) are compared in Figure 5 with the $\frac{h}{\ell} = 1.6$ correction included. It is seen that the discrepancy between the two is very small.

For h_o Graham and Rodriguez obtained

$$h_o = h\left/\left[\frac{1}{2} - \frac{\sum_{n=0}^{\infty}\left\{\frac{8 \tanh\{(2n+1)\frac{\pi}{2}\frac{h}{\ell}\}}{\pi^3(2n+1)^3\frac{h}{2\ell}}\left(\frac{1}{2} - \frac{\tanh\{(2n+1)\frac{\pi}{2}\frac{h}{\ell}\}}{\frac{\pi}{2}(2n+1)\frac{h}{2\ell}}\right)\right\}}{\left\{1 - \sum_{n=0}^{\infty}\left\{\frac{8 \tanh\{(2n+1)\frac{\pi}{2}\frac{h}{\ell}\}}{\pi^3(2n+1)^3\frac{h}{2\ell}}\right\}\right\}}\right]\right. \tag{18}$$

This is compared with the value of h_o given by equation (15') in Figure 6 where it is seen that the agreement is very close.

3. Rectangular Tank. Oscillating Fluid. The effect of the impulsive pressures is to excite the fluid into oscillations. To examine the fundamental mode of vibration consider the fluid to be constrained between rigid membranes that are free to rotate as shown in Figure 7. The constraint is described by the following equations.

$$u = \frac{\ell^2 - x^2}{2}\frac{d\dot{\theta}}{dy} \tag{19}$$

$$v = \dot{\theta} x \tag{20}$$

The pressure in the fluid is given by

$$\frac{\partial p}{\partial x} = -\rho \dot{u}$$

$$p = -\rho \frac{\ell^3}{2}\left(\frac{x}{\ell} - \frac{1}{3}\left(\frac{x}{\ell}\right)^3\right)\frac{d\ddot{\theta}}{dx} \tag{21}$$

The equation of motion of a slice of the fluid is

$$\int_{-\ell}^{+\ell} \frac{\partial p}{\partial y}\, dy \; x\, dx = -\rho \frac{(2\ell^3)}{12} \ddot{\theta}\, dy$$

or

$$\frac{d^2\ddot{\theta}}{dy^2} = \frac{5}{2}\frac{\ddot{\theta}}{\ell^2} \tag{22}$$

The solution of this equation, with the boundary conditions appropriate to the problem, is for sinusoidal oscillations

$$\theta = \theta_0 \frac{\sinh \sqrt{\frac{5}{2}}\, \frac{y}{\ell}}{\sinh \sqrt{\frac{5}{2}}\, \frac{h}{\ell}} \sin \omega t \tag{23}$$

This specifies the oscillation of the fluid. To determine the natural frequency of vibration the maximum kinetic energy, T, is equated to the maximum potential energy, V.

$$T = \int_0^h \int_{-\ell}^{+\ell} \frac{1}{2}\rho(u^2 + v^2)\, \omega^2 \sin^2 \omega t \;\, dx\, dy$$

$$V = \int_{-\ell}^{+\ell} \frac{1}{2}\rho g x^2 \sin \omega t \;\, dx$$

This gives

$$\omega^2 = \frac{g}{\ell} \sqrt{\frac{5}{2}}\, \tanh \sqrt{\frac{5}{2}}\, \frac{h}{\ell} \tag{24}$$

The third mode is found as shown in Figure 8 and similarly for the other modes. The circular frequencies are then for the nth mode.

$$\omega_n^2 = \frac{g}{\ell}\, n\sqrt{\frac{5}{2}}\, \tanh n\sqrt{\frac{5}{2}}\, \frac{h}{\ell} \tag{25}$$

The exact expression as given by Graham and Rodriguez is

$$\omega_n^2 = \frac{g}{\ell}\, n\frac{\pi}{2}\, \tanh n\frac{\pi}{2}\, \frac{h}{\ell}$$

Since $\frac{\pi}{2} = 1.57$ and $\sqrt{\frac{5}{2}} = 1.58$ there is good agreement.

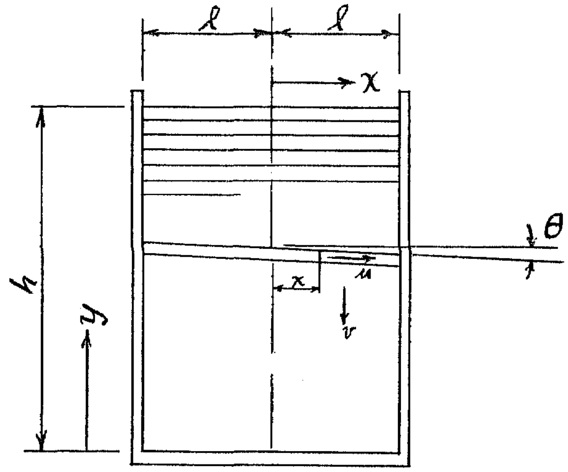

FIGURE 7

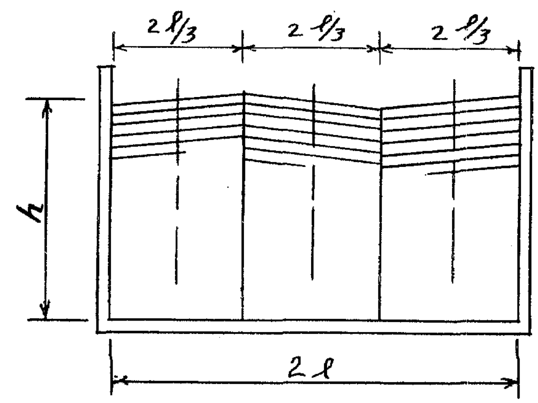

FIGURE 8

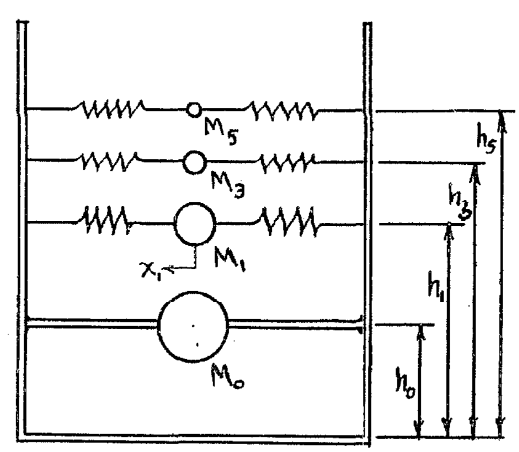

FIGURE 9

The pressure on the wall of the container is from (21)

$$p_w = \left(\rho \frac{\ell^3}{3} \sqrt{\frac{5}{2}} \; \frac{\cosh \sqrt{\frac{5}{2}} \frac{y}{\ell}}{\sinh \sqrt{\frac{5}{2}} \frac{h}{\ell}}\right) \omega^2 \theta_o \; \sin \omega t \tag{26}$$

The total force exerted on the wall is

$$P = \int_0^h p_w \, dy = \rho \frac{\ell^3}{3} \omega^2 \theta_o \; \sin \omega t \tag{27}$$

An equal force is exerted on the opposite wall. The total force of $2P$ may be considered to be produced by an equivalent mass, M_1, which is spring mounted as shown in Figure 9. The mass M_1 will oscillate and produce a horizontal force as follows:

$$x_1 = A_1 \sin \omega t$$

$$F_1 = -M_1 A_1 \, \omega^2 \sin \omega t$$

$$T = \frac{1}{2} M_1 A_1{}^2 \, \omega^2 \sin^2 \omega \, t$$

Comparing these equations with those for the fluid, we find

$$A_1 = \frac{\theta_o \, h}{\frac{h}{\ell} \sqrt{\frac{5}{2}} \tanh \sqrt{\frac{5}{2}} \frac{h}{\ell}} \tag{28}$$

$$M_1 = M \frac{1}{3} \sqrt{\frac{5}{2}} \frac{\ell}{h} \tanh \sqrt{\frac{5}{2}} \frac{h}{\ell} \tag{29}$$

This value of M_1 exceeds by less than 2% that given by Graham and Rodriguez.

The elevation of M_1 above the bottom of the tank is determined so that M_1 exerts the same moment as the fluid. If we consider only the moment exerted by the fluid on the walls (neglecting the fluid pressures on the tank bottom) we obtain

$$h_1 = h \left(1 - \frac{1}{\frac{h}{\ell} \sqrt{\frac{5}{2}} \tanh \sqrt{\frac{5}{2}} \frac{h}{\ell}} + \frac{1}{\frac{h}{\ell} \sqrt{\frac{5}{2}} \sinh \sqrt{\frac{5}{2}} \frac{h}{\ell}}\right) \tag{30}$$

If the pressures exerted on the bottom are also taken into account we obtain:

$$h_1 = h\left(1 - \frac{\cosh\sqrt{\frac{5}{2}}\frac{h}{\ell} - 2}{\sqrt{\frac{5}{2}}\frac{h}{\ell}\,\sinh\sqrt{\frac{5}{2}}\frac{h}{\ell}}\right) \tag{31}$$

The corresponding quantities for the higher modes are given by substituting $(\frac{\ell}{n})$ for ℓ, noting that only the modes having $n = 1, 3, 5, \ldots\ldots$ exert moments on the tank.

When the tank is subjected to an earthquake the various modes of vibration will be excited. The degree of excitation can be computed by replacing the fluid by the set of masses M_0, M_1, ... as shown in Figure 9 which thus reduces the problem to solving for the response of a number of simple oscillators. An elevated water tank can be treated in a similar fashion, the fluid merely introduces some additional degree of freedom.

4. Circular Tank. Impulsive Pressures. Consider a cylindrical tank as shown in Figure 10 and let the fluid be constrained between fixed membranes parallel to the x-axis, then each slice of unit thickness may be treated as if it were a narrow rectangular tank and the equations of the preceding sections will apply. The pressure exerted against the wall of the tank is, from equation (12)

$$p_w = -\rho \dot{u}_0 h\sqrt{3}\left(\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right)\tanh\left(\sqrt{3}\frac{R}{h}\cos\theta\right) \tag{32}$$

The pressure on the bottom of the tank is

$$p_b = -\rho \dot{u}_0 h\frac{\sqrt{3}}{2}\,\frac{\sinh\sqrt{3}\frac{x}{h}}{\cosh\sqrt{3}\frac{\ell}{h}} \tag{33}$$

where $\ell^2 = R^2 - x^2$

For tall narrow tanks, as shown in Figure 11, when $\frac{h}{R} > 1.6$ the fluid below depth h should be considered to move with the tank as a rigid body.

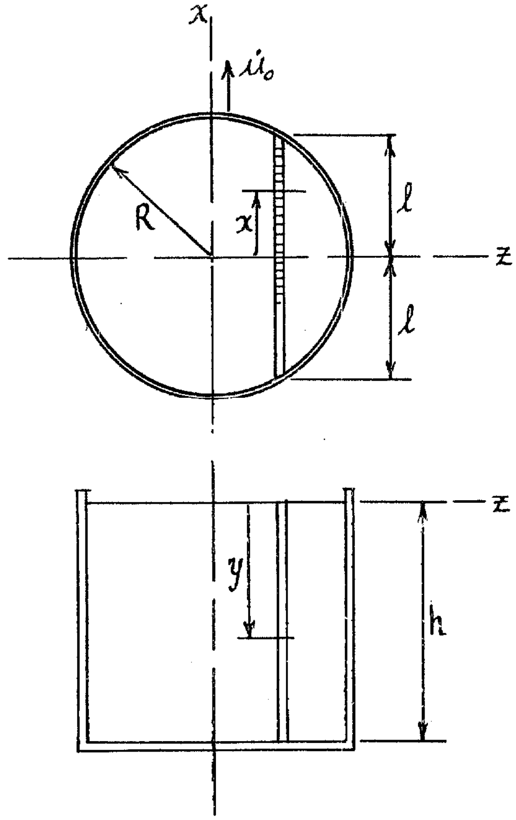

FIGURE 10

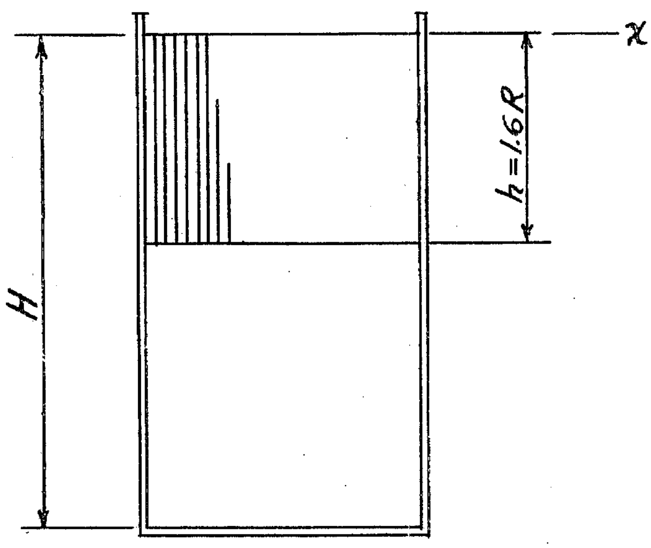

FIGURE 11

The preceding expressions are not convenient for calculating the total force exerted by the fluid. The following modification gives very accurate values for $\frac{R}{h}$ small and somewhat overestimates the pressure for $\frac{R}{h}$ large.

$$p_w = -\rho \ddot{u}_0 h\sqrt{3}\, \tanh\sqrt{3}\frac{R}{h}\left\{\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right\}\cos\theta \tag{32'}$$

From this expression the total force exerted on the walls is

$$\int_0^h\int_0^{2\pi} p_w \cos\theta\, R\,d\theta\,dy = -\rho \ddot{u}_0 \pi R^2 h \;\frac{\tanh\sqrt{3}\frac{R}{h}}{\sqrt{3}\frac{R}{h}} \tag{33'}$$

from which it is seen that the force exerted is the same as if an equivalent mass M_o were moving with the tank, where

$$M_o = M\frac{\tanh\sqrt{3}\,R/h}{\sqrt{3}\,R/h} \tag{34}$$

This expression is compared in Figure 12 with that computed by Jacobsen (1949) and it is seen that the agreement is very close.

To exert a moment equal to that of the fluid pressure on the wall the equivalent mass M_o should be at a height above the bottom ($\frac{h}{R} \leq 1.6$)

$$h_o = h\,\frac{3}{8} \tag{35}$$

If the moment due to the pressures exerted on the bottom of the tank *are* included the equivalent mass M_o must be at a height

$$h_o = \frac{3}{8}h\left(1 + \frac{4}{3}\left(\frac{\sqrt{3}\,R/h}{\tanh\sqrt{3}\,R/h} - 1\right)\right) \tag{36}$$

to produce the proper total moment on the tank. This agrees well with that computed by Jacobsen as shown in Figure 13.

5. <u>Circular Tank. Oscillating Fluid.</u> To examine the first mode of vibration of the fluid consider constraints to be provided by horizontal

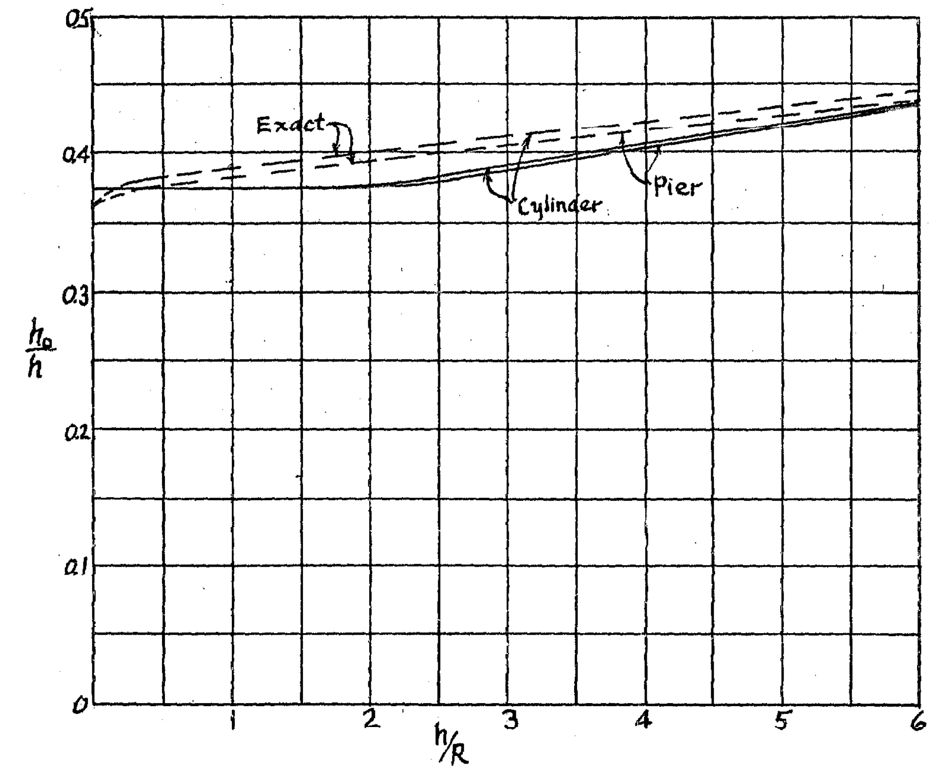

FIGURE 13

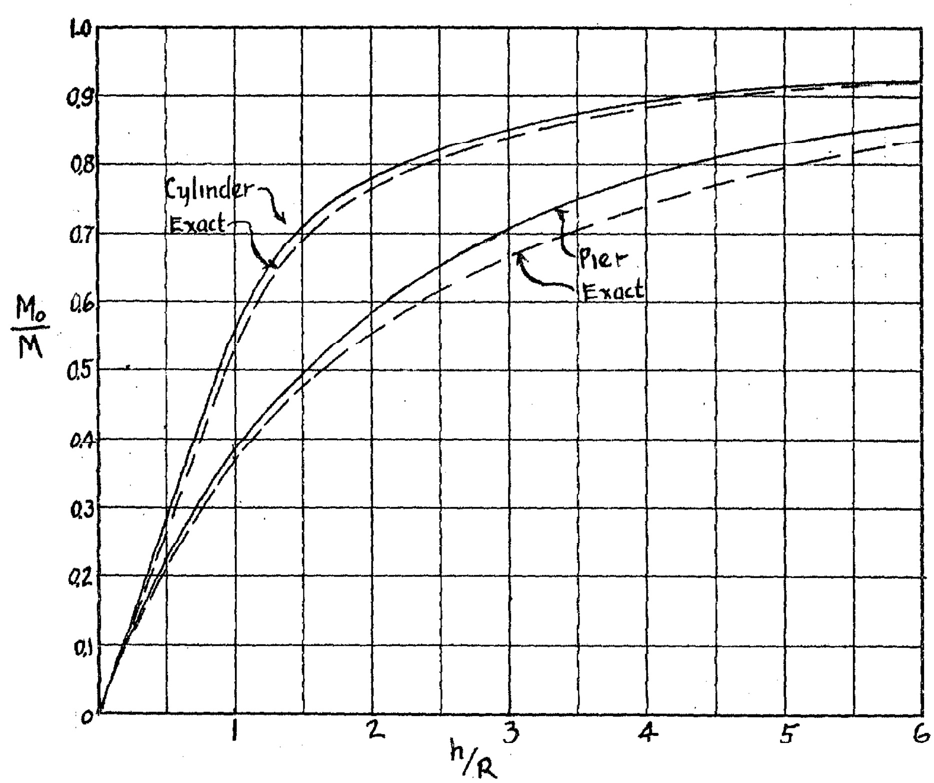

FIGURE 12

membranes free to rotate, as shown in Figure 14. Let u, v, w, be the x, y, z components of velocity and describe the constraints on the flow by the following equations

$$\frac{\partial(ub)}{\partial x} = -b\frac{\partial v}{\partial y} \qquad (37)$$

$$v = x\dot{\theta} \qquad (38)$$

$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (39)$$

These equations state that all the fluid at a given x, y moves with the same velocity v, and the fluid at a given x moves with a uniform u. From these equations we have

$$u = -\frac{1}{b}\frac{\partial\dot{\theta}}{\partial y}\int_{-R}^{x} x\, b\, dx$$

$$w = z\frac{b'}{b^2}\frac{\partial\dot{\theta}}{\partial y}\int_{-R}^{x} x\, b\, dx$$

where $b' = \frac{db}{dx}$

The total kinetic energy is thus

$$T = \frac{1}{2}\rho\int_0^{h}\int_{-R}^{+R}\int_{-b}^{+b}\left\{x^2\dot{\theta}^2 + \frac{1}{b^2}\left(\frac{\partial\dot{\theta}}{\partial y}\right)^2\left(\int_{-R}^{x} xb\, dx\right)^2\left(1 + z^2\frac{b'^2}{b^2}\right)\right\} dx\, dy\, dz$$

$$= \frac{1}{2}\rho\int_0^{h}\left\{I_z\dot{\theta}^2 + K\left(\frac{\partial\dot{\theta}}{\partial y}\right)^2\right\} dy$$

where $I_z = \int_A x^2 dA \qquad K = 2\int_{-R}^{+R}\frac{1}{b}\left(\int_{-R}^{x} b\, x\, dx\right)^2\left(1 + \frac{b'^2}{3}\right) dx$

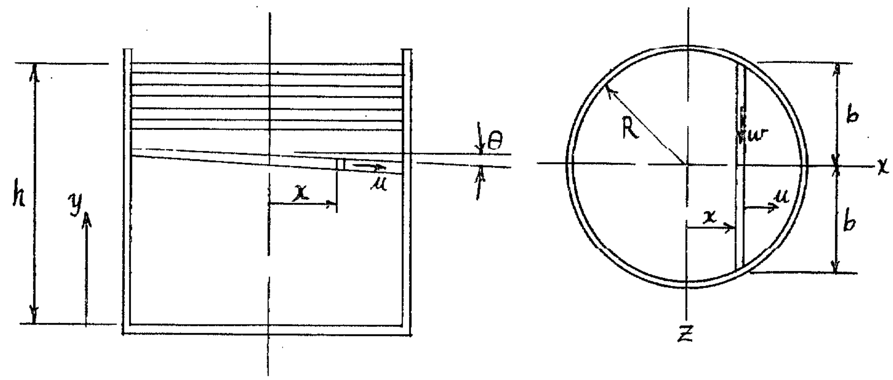

FIGURE 14

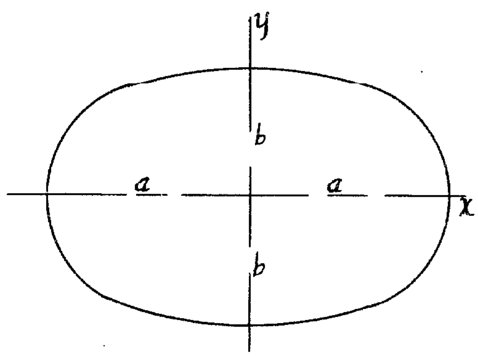

FIGURE 15

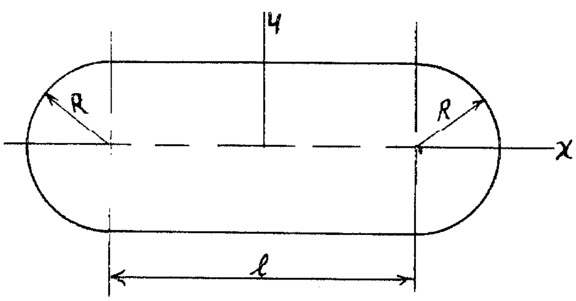

FIGURE 16

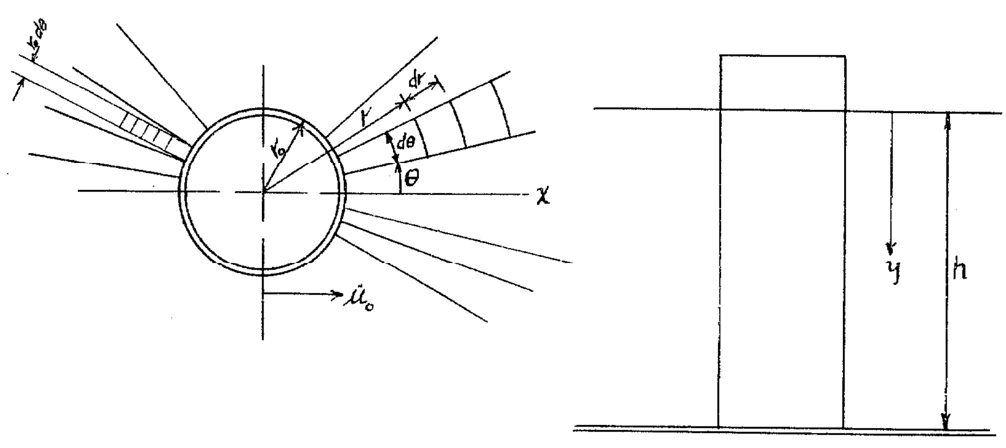

FIGURE 17

The potential energy of the fluid is

$$V = \frac{1}{2} g\rho\, \theta_h^2 \int x^2\, dx\, dz = \frac{1}{2} g\rho\, \theta_h^2 I_z$$

By Hamilton's Principle

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0$$

$$\delta \int_{t_1}^{t_2} \frac{1}{2}\rho \left\{ \int_0^h \left[I_z \dot{\theta}^2 + \left(\frac{\partial \dot{\theta}}{\partial y} \right)^2 K \right] dy - g\, \theta_h^2 I_z \right\} dt$$

or $$\int_{t_1}^{t_2} \int_0^h \rho \left(I_z \ddot{\theta} - \frac{\partial^2 \ddot{\theta}}{\partial y^2} \right) \delta\theta\, dx\, dt + \int_{t_1}^{t_2} \rho \left(K^2 \left(\frac{\partial \ddot{\theta}}{\partial y} \right)_h + g\, I_z\, \theta_h \right) \delta\theta_h\, dt = 0$$

This gives the two equations

$$\frac{\partial^2 \ddot{\theta}}{\partial y^2} - \frac{I_z}{K} \ddot{\theta} = 0 \tag{40}$$

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial \theta}{\partial y} \right)_h + g I_z \theta_h = 0 \tag{41}$$

From which we obtain for free vibrations

$$\theta = \theta_0 \frac{\sinh \sqrt{\frac{I_z}{K}}\, y}{\sinh \sqrt{\frac{I_z}{K}}\, h} \sin \omega t \tag{42}$$

$$\omega^2 = g \sqrt{\frac{I_z}{K}} \tanh \sqrt{\frac{I_z}{K}}\, h \tag{43}$$

It will be observed that this analysis is quite general and applies to any cylindrical container for which the x, y axes are axes of symmetry. For the circular tank

$$I_z = \frac{\pi R^4}{4} \qquad K = \frac{2\pi}{27} R^6$$

$$\omega^2 = \frac{g}{R}\sqrt{\frac{27}{8}} \tanh\left(\sqrt{\frac{27}{8}}\frac{h}{R}\right) \tag{44}$$

The exact solution as given in Lamb's "Hydrodynamics" is

$$\omega^2 = \frac{g}{R}\, 0.586\,\pi \tanh\left(0.586\,\pi \frac{h}{R}\right)$$

which agrees within 1% with the approximate solution.

The pressure in the fluid is given by

$$\frac{\partial p}{\partial z} = -\rho \dot{w} \qquad \frac{\partial p}{\partial x} = -\rho \dot{u}$$

$$p = -\rho \frac{\partial \ddot{\theta}}{\partial y}\left\{\int \frac{Q}{b}\, dx\right\} \tag{45}$$

$$Q = \int_{-R}^{x} bx\, dx$$

For the circular tank this is

$$p = -\rho \frac{\partial \ddot{\theta}}{\partial y} \frac{R^2}{3}\left\{\frac{x}{R} - \frac{1}{3}\left(\frac{x}{R}\right)^3\right\}$$

$$\frac{\partial \ddot{\theta}}{\partial y} = -\left(\sqrt{\frac{27}{8}}\frac{1}{R}\frac{\cosh\sqrt{\frac{27}{8}}\frac{y}{R}}{\sinh\sqrt{\frac{27}{8}}\frac{h}{R}}\right)\theta_0\,\omega^2 \sin\omega t \tag{46}$$

The pressure on the wall is

$$p_w = -\rho \frac{\partial \ddot{\theta}}{\partial y}\frac{R^2}{3}\left(1 - \frac{\cos^2\theta}{3}\right)\cos\theta \tag{47}$$

The resultant horizontal force exerted on the wall is

$$P = -\pi\frac{1}{4}\,\rho\,\omega^2 R^4 \theta_0 \sin\omega t \tag{48}$$

This force may be considered to be produced by an equivalent mass M_1 (see Figure 9) oscillating in a horizontal plane with motion

$$x_1 = A_1 \sin \omega t$$

$$M_1 = M \frac{1}{4}\sqrt{\frac{27}{8}} \; \frac{R}{h} \tanh\sqrt{\frac{27}{8}}\frac{h}{R} \tag{49}$$

$$A_1 = \theta_0 h \frac{1}{\sqrt{\frac{27}{8}}\frac{h}{R} \tanh\sqrt{\frac{27}{8}}\frac{h}{R}} \tag{50}$$

In order that M_1 exert the same moment as the fluid pressures on the wall, it should be at an elevation above the bottom of

$$h_1 = h\left(1 - \frac{1}{\sqrt{\frac{27}{8}}\frac{h}{R} \tanh\sqrt{\frac{27}{8}}\frac{h}{R}} + \frac{1}{\sqrt{\frac{27}{8}}\frac{h}{R} \sinh\sqrt{\frac{27}{8}}\frac{h}{R}}\right) \tag{51}$$

The pressure exerted on the bottom of the tank is

$$p_b = -\rho\omega^2\sqrt{\frac{3}{8}} \frac{R^2}{\sinh\sqrt{\frac{27}{8}}\frac{h}{R}} \left\{\frac{x}{R} - \frac{1}{3}\left(\frac{x}{R}\right)^3\right\}\theta_0 \tag{52}$$

This exerts a moment about the z-axis equal to

$$\frac{10}{48}\sqrt{\frac{3}{8}} \; \frac{\pi R^5 \rho\omega^2}{\sinh\sqrt{\frac{27}{8}}\frac{h}{R}}$$

Including this, the correct total moment on the tank is produced when

$$h_1 = h\left(1 - \frac{2\cosh\sqrt{\frac{27}{8}}\frac{h}{R} - \frac{31}{16}}{\sqrt{\frac{27}{8}}\frac{h}{R} \sinh\sqrt{\frac{27}{8}}\frac{h}{R}}\right) \tag{53}$$

6. <u>Elliptical Tank.</u> For the elliptical tank, as shown in Figure 15, the impulsive pressure on the wall is given by equation (12)

$$p_w = \rho \dot{u}_0 h\left(\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right)\sqrt{3}\tanh\sqrt{3}\frac{\ell}{h} \tag{54}$$

with a similar expression for acceleration in the direction of the y-axis.

For oscillations of the fluid, equations (37) through (43) apply and we obtain for the first mode, about the y-axis

$$\omega^2 = \frac{g}{a}\sqrt{\frac{54}{15+\left(\frac{b}{a}\right)^2}} \tanh \sqrt{\frac{54}{15+\left(\frac{b}{a}\right)^2}}\, \frac{h}{a} \tag{55}$$

For $\frac{h}{a}$ small this reduces to

$$\omega^2 = \frac{\sqrt{gh}}{a} A \qquad\qquad A = \sqrt{\frac{54}{15+\left(\frac{b}{a}\right)^2}}$$

and comparing with the exact solution (Jeffrey, 1924) the following values are obtained

$\frac{b}{a}$	A (Approx.)	A (Exact)
1	1.84	1.84
0.6	1.88	1.87
0	1.90	1.89

7. Composite Tanks. Symmetrical tanks formed of composite shapes such as that shown in Figure 16 will have impulsive pressures given by equation (12) and oscillations as described by equations (37) and (43). The tank shown in Figure 16 has

$$I_y = \frac{2R\ell^3}{12} + \pi R^4\left(\frac{1}{4} + \left(\frac{\ell}{2R} + \frac{4}{3\pi}\right)^2\right)$$

$$K_y = R\ell^5\left\{0.233\left(\frac{R}{\ell}\right)^5 + 0.627\left(\frac{R}{\ell}\right)^4 + 1.377\left(\frac{R}{\ell}\right)^3 + 0.197\left(\frac{R}{\ell}\right)^2 + 0.131\frac{R}{\ell} + 0.0166\right\} \tag{56}$$

8. Circular Cylinder Surrounded by Fluid. Consider the rigid cylinder shown in Figure 17 with flow constrained by radial membranes and by vertical membranes at r and r + dr from the center. The vertical velocity is given by

$$v = (h-y)\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) \tag{57}$$

The kinetic energy of fluid in a filament of width* $r_o \, d\theta$

$$T = \iint \tfrac{1}{2}\rho(u^2+v^2) r_o \, dy \, dr \, d\theta$$
$$= \iint \tfrac{1}{2}\rho\left\{(h-y)^2\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)^2 + u^2\right\} r_o \, dy \, dr \, d\theta$$

Applying Hamilton's Principle yields the equation

$$\frac{\partial^2 u}{\partial r^2} - \left(\frac{2}{r^2}+\frac{3}{h^2}\right)u = 0 \tag{58}$$

The solution is

$$u = u_o \, e^{-\sqrt{3}\frac{r-r_o}{h}} \left\{ \frac{\frac{h}{r}+\sqrt{3}}{\frac{h}{r_o}+\sqrt{3}} \right\} \tag{59}$$

The pressure in the fluid is given by

$$\frac{\partial p}{\partial y} = -\rho \dot{v}$$
$$p = -\rho \dot{u}_o h\left(\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right) 3 \, \frac{e^{-\sqrt{3}\frac{r-r_o}{h}}}{\frac{h}{r_o}+\sqrt{3}} \tag{60}$$

If the cylinder is given an acceleration $\dot{U}_o$ in the x - direction then

$$\dot{u}_o = \dot{U}_o \cos\theta$$

and the pressure on the cylinder is

$$p_w = -\rho \dot{U}_o \, \frac{3h\left(\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right)}{h/r_o + \sqrt{3}} \cos\theta \tag{61}$$

The resultant force on the cylinder in the x - direction is

$$P = \int_0^{2\pi}\int_0^{h} p_w \, r_o \cos\theta \, d\theta \, dy$$
$$= -\rho \dot{U}_o \pi r_o^2 h \, \frac{h/r_o}{\frac{h}{r_o}+\sqrt{3}} \tag{62}$$

---

* As seen in Figure 17, this imposes a special constraint on the flow. If this artifice is not used one obtains a Bessel function of the first order, imaginary argument (K_1), instead of Equation (59), which is a very good approximation to K_1.

The moment exerted on the cylinder at height y, is

$$\text{Mom.} = \frac{1}{2}\rho \dot{U}_0 \frac{\pi r_0}{\frac{h}{r_0}+\sqrt{3}} \; y^3\left(1-\frac{1}{4}\frac{y}{h}\right) \tag{63}$$

From (61) and (62) it is seen that the action of the fluid is the same as if an equivalent mass M_o were rigidly fastened to the cylinder at h_o, where

$$\frac{M_0}{\rho\pi r_0^2 h} = \frac{h/r_0}{h/r_0+\sqrt{3}} \tag{64}$$

$$h_0 = \frac{3}{8}h \tag{65}$$

The preceding formulas should be used only for $h/r_o \leqq 1.6$. When h/r_o exceeds this the fluid below the level $y/r_o = 1.6$ should be considered to have the regular two-dimensional flow past the cylinder for which

$$p_w = \rho \dot{U}_0 r_0 \cos\theta \tag{66}$$

and the force for unit length of cylinder is

$$\text{P per ft.} = -\rho \dot{U}_0 \pi r_0^2$$

For this range, $\left(\frac{h}{R} > 1.6\right)$,

$$\begin{aligned} P &= -\rho \dot{U}_0 \pi r_0 h\left(1-0.832\frac{r_0}{h}\right) \\ h_0 &= \frac{h}{2}\left\{\frac{1-1.66\,r_0/h+1.03\,(r_0/h)^2}{1-0.832\frac{r_0}{h}}\right\} \end{aligned} \tag{67}$$

A comparison of the foregoing approximate values for M_o and h_o with Jacobsen (1949) are shown in Figures 12 and 13.

9. <u>Rectangular Dam</u>. For the rectangular dam with sloping face, as shown in Figure 18, the impulsive pressures are given by the following equations:

$$
\begin{aligned}
v &= (h-y)\frac{du}{dx} + u\cos\phi \\
u &= u_o\, e^{-\sqrt{3}\, x/h} \\
p_w &= \rho\, \dot{u}_o\, h\left\{\left(\frac{y}{h}-\frac{1}{2}\left(\frac{y}{h}\right)^2\right)\sqrt{3} - \frac{y}{h}\cos\phi\right\} \\
P_w &= \rho\, \ddot{u}_o\, h^2\left\{\frac{1}{\sqrt{3}} - \frac{\cos\phi}{2}\right\}
\end{aligned}
\tag{68}
$$

The resultant horizontal force on the face of the dam is

$$F_H = \rho\, \dot{u}_o\, H^2 \frac{1}{\sin\phi}\left\{\frac{1}{\sqrt{3}} - \frac{\cos\phi}{2}\right\} \tag{69}$$

When the face of the dam is vertical $F_H = 0.577 \; \rho \dot{u}_o H^2$ which is slightly larger than the $0.543 \; \rho \dot{u}_o H^2$ given by Westergaard (1933). Equations (68) and (69) are suitable only when $\phi > 45^o$ (see Figure 20). For $\phi < 45^o$ a different approximation must be used, as given below.

When $\phi < 45^o$ the fluid should be divided into two regions as shown in Figure 19, where a rigid membrane lies along the x-axis and has a horizontal acceleration $c\,\dot{u}_o$ such that the pressure force on each side of the membrane is the same. In the region to the left of the x-axis the following equations describe the flow.

$$
\begin{aligned}
v &= (h-y)\frac{\partial u}{\partial x} + c\, u_o \cos\phi \\
\frac{\partial p}{\partial y} &= -\rho v
\end{aligned}
\tag{70}
$$

Applying Hamilton's Principle to the total kinetic energy in this region leads to the equation

$$x^2\frac{d^2u}{dx^2} + 3x\frac{du}{dx} - 3\tan^2\phi\; u = 3c\,\dot{u}_o \cos\phi\tan\phi \tag{71}$$

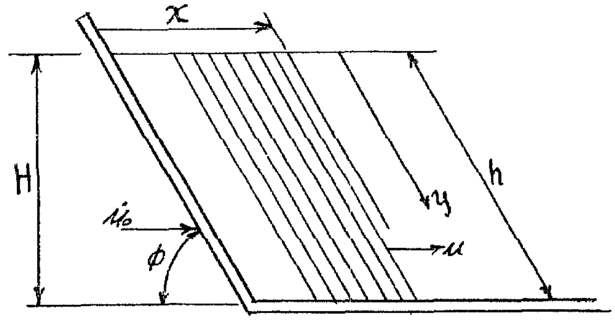

FIGURE 18

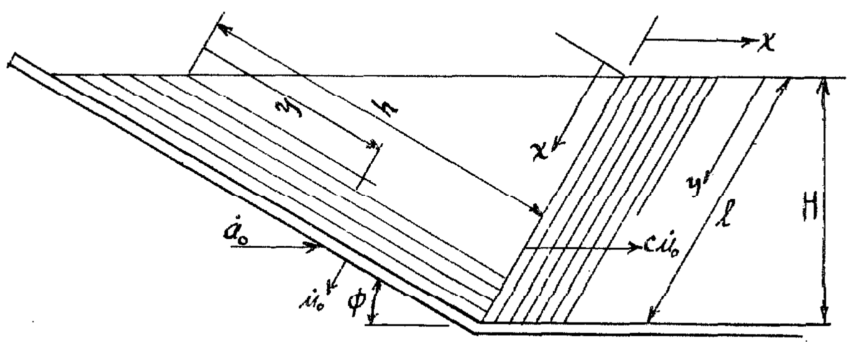

FIGURE 19

The appropriate solution is

$$u = u_0\left(1 - c\,\frac{\cos^2\phi}{\sin^2\phi}\right)\left(\frac{x}{\ell}\right)^{\alpha} + c\,\frac{\cos^2\phi}{\sin\phi}$$

$$\alpha = \sqrt{1+3\tan^2\phi}\ -1$$

The pressure in the fluid is

$$p = -\rho u_0^2\left\{\left(hy - \tfrac{1}{2}y^2\right)\frac{\alpha}{\ell}\left(1 - c\,\frac{\cos^2\phi}{\sin\phi}\right)\left(\frac{x}{\ell}\right)^{\alpha-1} + c\,y\cos\phi\right. \qquad (72)$$

The total pressure force on the membrane along the x-axis is

$$P = \int_0^{\ell} p_{y=\ell}\, dx = -\rho u_0^2\left\{\frac{\ell^2}{2\tan^2\phi}\,\frac{\alpha}{\alpha+2}\left(1 - c\,\frac{\cos^2\phi}{\sin\phi}\right)\left(\frac{x}{\ell}\right)^{\alpha+1} + \frac{\ell^2}{2\tan^2\phi}\,c\cos\phi\left(\frac{x}{\ell}\right)^2\right\}_0^{\ell} \qquad (73)$$

In the region to the right of the x-axis

$$v = (\ell - y)\frac{du}{dx} + u\cos\phi$$

$$p = -\rho\left(\ell y - \tfrac{1}{2}y^2\right)\frac{du}{dx} + y\,u\cos\phi$$

$$P = \quad \rho\left\{\frac{\ell^3}{3}\frac{du}{dx} + \frac{\ell^2}{2}u\cos\phi\right\}$$

The equation of motion is

$$\frac{d^2u}{dx^2} - \frac{3}{h^2}u = 0$$
$$u = u_0\, e^{-\sqrt{3}\frac{x}{h}} \qquad (74)$$

The normal force at $x = 0$ is

$$P = \rho u_0^2 \ell^2\left\{\frac{1}{\sqrt{3}} - \frac{\cos\phi}{2}\right\}$$

Equating this to equation (73) and solving for C, gives

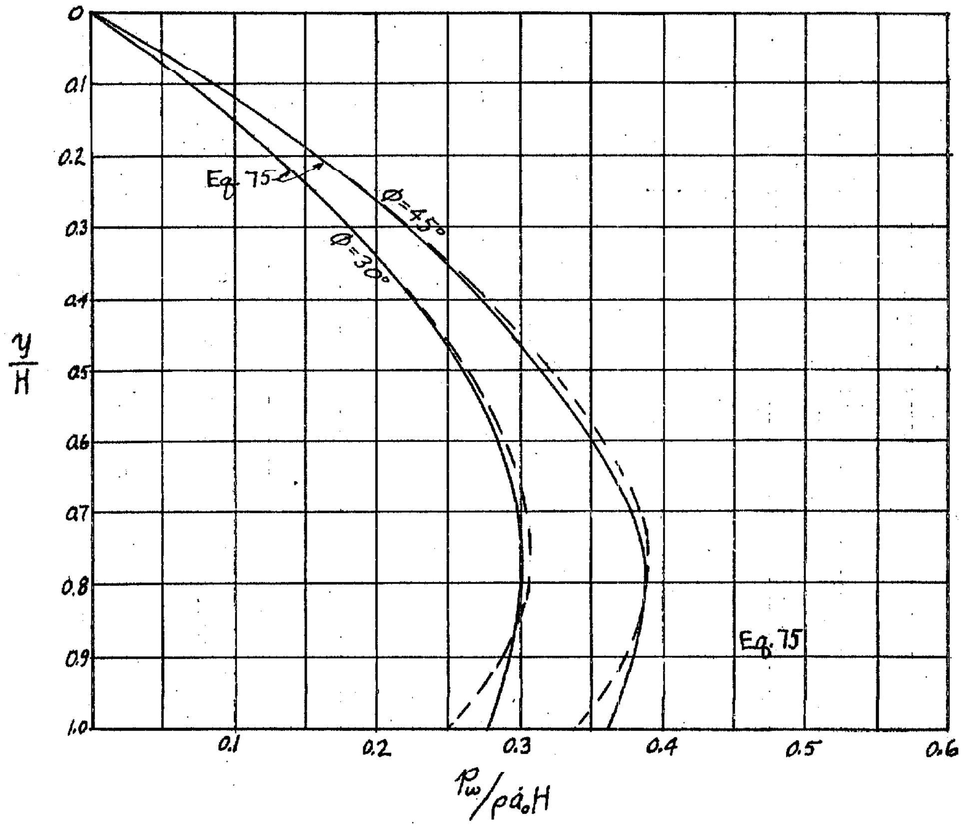

FIGURE 20

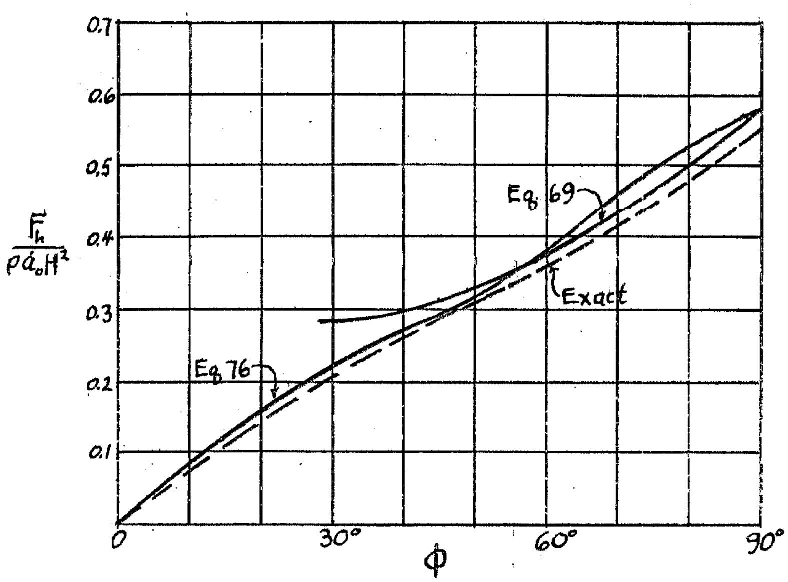

FIGURE 21

$$C = -\frac{\frac{\alpha}{\alpha+2}}{2\beta\tan^2\phi + \cos\phi - \frac{\cos^2\phi}{\sin\phi}\,\frac{\alpha}{\alpha+2}}$$

$$\beta = \left(\frac{1}{\sqrt{3}} - \frac{\cos\phi}{2}\right)$$

The pressure on the inclined face is thus

$$p_w = -\rho\,\dot{a}_o H\left\{\left(\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^2\right)\frac{\alpha}{\tan\phi}\left(1 - c\,\frac{\cos^2\phi}{\sin\phi}\right) + c\,\frac{y}{h}\cos\phi\right\} \qquad (75)$$

where $\dot{a}_o$ is the horizontal acceleration of the inclined face. The resultant horizontal force exerted against the inclined face is

$$F_H = -\rho\,\dot{a}_o H^2\left\{\frac{\alpha}{\tan\phi}\left(1 - c\,\frac{\cos^2\phi}{\sin\phi}\right) + \frac{c}{2}\cos\phi\right\} \qquad (76)$$

Equations (75) and (76) are compared in Figures 20 and 21 with the corresponding quantities calculated by the relaxation method. It is seen that the agreement is good except that the pressures are overestimated at the toe.

10. Trapezoidal Dam. For a trapezoidal shape as shown in Figure 22 the formulas of the preceding section (with variable h) may be used for the pressures at various sections across the width of the dam. This will somewhat overestimate the force on the dam. For narrow wedge-shaped forms the error increases and better results are obtained by the following procedure. Consider the constraints on the flow to be as shown in Figure 23 and as described by the following equations:

$$u = u(x,t) \qquad w = w(x,t)$$

$$w = -\frac{1}{b}\,\frac{\partial u}{\partial x}\int_0^z b\,dz - u\cos\phi \qquad (77)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

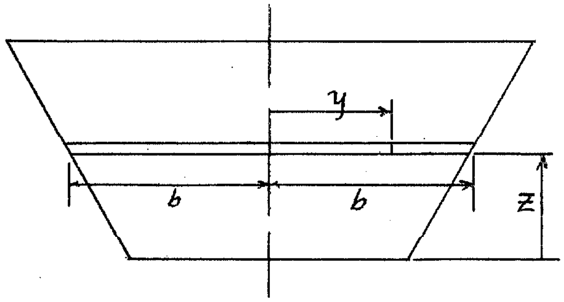

FIGURE 22

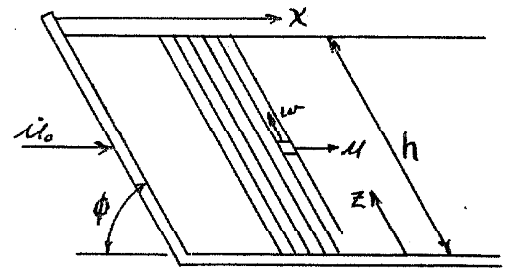

FIGURE 23

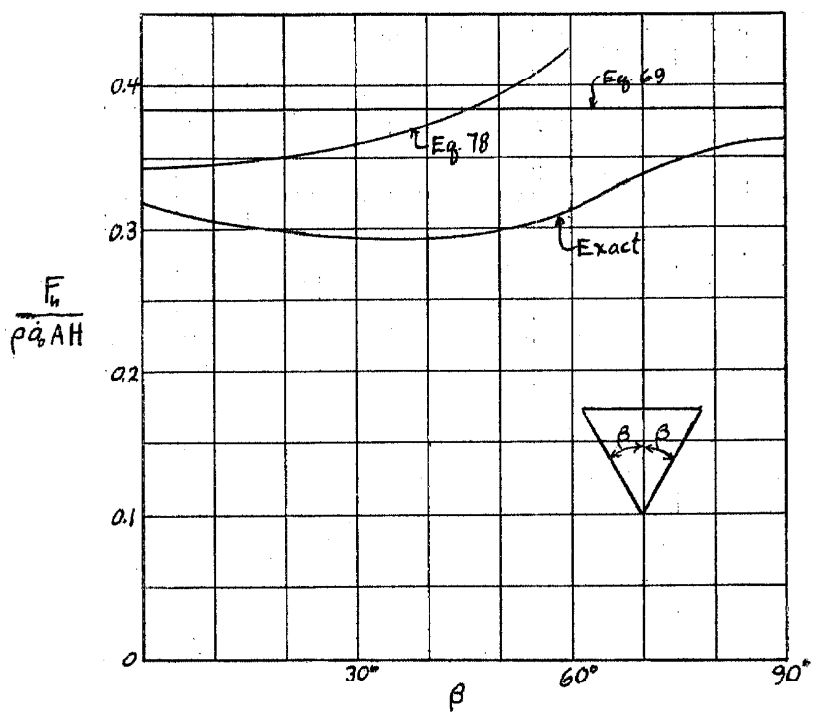

FIGURE 24
Vertical triangular dam

$$v = y \frac{du}{dx} \frac{b'}{b^2} \int_0^z b\, dz$$
$$b' = \frac{db}{dx} \qquad (77)$$

Applying Hamilton's Principle to the kinetic energy gives:

$$\frac{d^2u}{dx^2} - \frac{A}{K} u = 0$$

A = area of dam face

$$K = 2 \int_0^h \frac{\left(\int_0^z b\,dz\right)^2}{b} \left\{1 + \frac{b'^2}{3}\right\} dz$$

$$u_\phi = u_0\, e^{-\sqrt{\frac{A}{K}}\, x}$$

The pressure on the face is

$$p_w = -\rho \dot{u}_0 \sqrt{\frac{A}{K}} \left\{ \int_z^h \frac{1}{b} \int_0^z b\, dz + \frac{y^2}{2} \frac{b'}{b^2} \int_0^z b\, dz + \sqrt{\frac{K}{A}} (z-h) \cos\phi \right\} \qquad (78)$$

The resultant horizontal force exerted on the dam is

$$F_H = -\rho \dot{u}_0 \frac{1}{\sin\phi} \left\{ \sqrt{AK} + \int_0^h (z-h)\, 2b\, dz \cos\phi \right\} \qquad (79)$$

For the triangle this is,

$$F_H = -\rho \dot{u}_0 \frac{h^3 \tan\beta}{\sin\phi} \left\{ \sqrt{\frac{1}{8}\left(1 + \frac{\tan^2\beta}{3}\right)} - \frac{\cos\phi}{3} \right\}$$

Figure 24 shows the force on a vertical triangular face as determined by (79) and by the application of (69), together with the value given by the exact solution. From this one can estimate the error involved in applying the equations to trapezoidal sections. Equations (78) and (79) are applicable only for $\phi > 45^\circ$

11. Stepped Dam. A stepped dam as shown in Figure 25 may be treated by applying equations similar to equations (1) to (4). In this case there are three regions of the fluid and the flow is described by

$$\bar{v}_1 = (h_1 - y)\frac{\partial \bar{u}_1}{\partial x} \qquad \text{in A}$$

$$v_1 = (h_1 - y_1)\frac{\partial u_1}{\partial x} + h_2\frac{\partial u_2}{\partial x} \qquad \text{in B} \qquad (80)$$

$$v_2 = (h_2 - y_2)\frac{\partial u_2}{\partial x} \qquad \text{in C}$$

These lead to the following equations of motion.

$$\frac{h_1^3}{3}\frac{d^2\bar{u}_1}{dx^2} - h_1\bar{u}_1 = 0$$

$$\frac{h_1^3}{3}\frac{d^2u_1}{dx^2} + \frac{h_2h_1^2}{2}\frac{d^2u_2}{dx^2} - h_1u_1 = 0$$

$$\frac{h_2^3}{3}\frac{d^2u_2}{dx^2} + h_1h_2^2\frac{d^2u_2}{dx^2} + \frac{h_1h_2^2}{2}\frac{d^2u_1}{dx^2} - h_2u_2 = 0$$

$$\bar{u}_1 = u_0 \text{ at } \bar{x}_1 = 0; \quad \bar{u} = u_1 \text{ at } x = 0 \qquad (81)$$

$$\frac{h_1^3}{3}\frac{d\bar{u}_1}{dx} = \frac{1}{2}h_1^2h_2\frac{du_2}{dx} + \frac{h_1^3}{3}\frac{du_1}{dx} \quad \text{at } x = 0$$

$$u_2 = u_0 \quad \text{at } x = 0$$

The pressures and forces on the faces of the dam are given by

$$\bar{p}_1 = -\rho(h_1y_1 - \tfrac{1}{2}y_1^2)\frac{d\dot{\bar{u}}_1}{dx} \qquad \text{in A}$$

$$\bar{P}_1 = -\rho\frac{h_1^3}{3}\frac{d\dot{\bar{u}}_1}{dx}$$

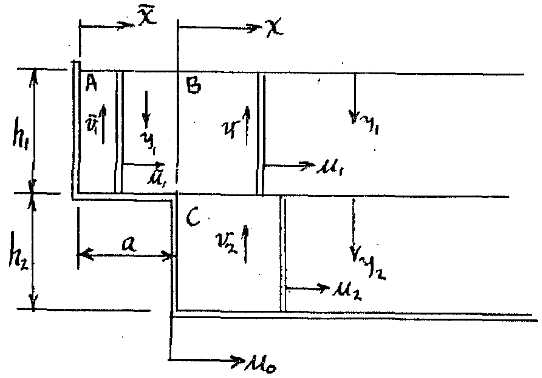

FIGURE 25

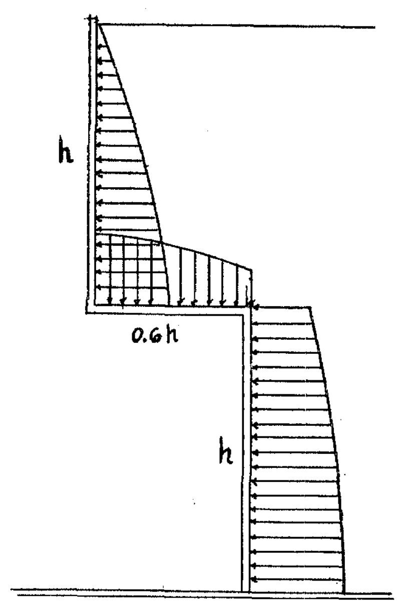

FIGURE 26
Stepped dam

$$p_1 = -\rho\left\{ h_2 y_1 \frac{d\dot{u}_2}{dx} + \left(h_1 y_1 - \tfrac{1}{2} y_1^2\right) \frac{d\dot{u}_1}{dx} \right.$$

$$P_1 = -\rho\left\{ \frac{h_1^2 h_2}{2} \frac{d\dot{u}_2}{dx} + \frac{h_1^3}{3} \frac{d\dot{u}_1}{dx} \right\} \qquad \text{in B}$$

$$p_2 = -\rho\left\{ h_2 h_1 \frac{d\dot{u}_2}{dx} + \frac{h_1^2}{2} \frac{d\dot{u}_1}{dx} + \left(h_2 y_2 - \tfrac{1}{2} y_2^2\right) \frac{d\dot{u}_2}{dx} \right.$$

$$P_2 = -\rho\left\{ h_2^2 h_1 \frac{d\dot{u}_2}{dx} + h_1^2 h_2 \frac{d\dot{u}_1}{dx} + \frac{h_1^3}{3} \frac{d\dot{u}_2}{dx} \right\} \qquad \text{in C}$$

In the particular case where $h_1 = h_2 = h$ we obtain

$$\bar{u}_1 = u_0\, e^{-\sqrt{3}\frac{x}{h}} - 2A_2 \sinh\sqrt{3}\frac{x}{h}$$

$$u_1 = A_1\left(0.423\, e^{-0.812\frac{x}{h}} + 2.42\, e^{-2.81\frac{x}{h}}\right) - 2.42\, u_0\, e^{-2.81\frac{x}{h}}$$

$$u_2 = A_2\left(e^{-0.812\frac{x}{h}} - e^{-2.81\frac{x}{h}}\right) + u_0\, e^{-2.81\frac{x}{h}}$$

where $\quad 2A_2 \sinh\sqrt{3}\frac{a}{h} + 2.84 A_1 = u_0\left(e^{\sqrt{3}\frac{a}{h}} + 2.42\right)$

$$2A_2 \cosh\sqrt{3}\frac{a}{h} - 2.39 A_1 = u_0\left(e^{\sqrt{3}\frac{a}{h}} - 1.49\right)$$

The following numerical values are obtained for the horizontal forces:

$\frac{a}{h}$	$\frac{\overline{P}_1}{\rho \dot{M}_o h^2}$	$\frac{P_2}{\rho \dot{M}_o h^2}$	$\frac{\overline{P}_1 + P_2}{\rho \dot{M}_o (2h)^2}$
0	0.805	1.449	0.564
0.2	0.752	1.312	0.512
0.4	0.707	1.186	0.478
0.6	0.681	1.16	0.460
0.8	0.657	1.12	0.445
∞	0.577	0.936	

A graph of the pressures is shown in Figure 26.

12. Segmental Dam. If the dam face is stepped as shown in Figure 29, and the angle ϕ is not less than 45° satisfactory results are obtained by the following procedure.

$$v_1 = (h_1 - y_1)\frac{du_1}{dx} + u_1 \cos\phi$$

$$v_2 = (h_2 - y_2)\frac{du_2}{dx} + h_1 \frac{du_1}{dx} \sin\phi$$

$$p_1 = -\rho\Big\{\Big(h_1 y_1 - \frac{y_1^2}{2}\Big)\frac{d\dot{u}_1}{dx} + \dot{u}_1 y_1 \cos\phi + \frac{h_2^2}{2}\frac{d\dot{u}_2}{dx} + h_1 h_2 \sin\phi \frac{d\dot{u}_1}{dx}$$

$$p_2 = -\rho\Big\{\Big(h_2 y_2 - \frac{y_2^2}{2}\Big)\frac{d\dot{u}_2}{dx} + h_1 y_2 \sin\phi \frac{d\dot{u}_1}{dx}\Big\}$$

These lead to the following equations of motion

$$\frac{d^2\dot{u}_1}{dx^2} - A\dot{u}_1 = -B\frac{d^2\dot{u}_2}{dx^2}$$

$$\frac{d^2\dot{u}_2}{dx^2} - C\dot{u}_2 = -D\frac{d^2\dot{u}_1}{dx^2}$$

$$A = \frac{3}{h_1^2\left(1+3\frac{h_2}{h_1}\sin\phi\right)} \qquad B = \frac{\frac{3}{2}\left(\frac{h_2}{h_1}\right)^2}{1+3\frac{h_2}{h_1}\sin\phi}$$

$$C = \frac{3}{h_2^2} \qquad D = \frac{3}{2}\frac{h_1}{h_2}\sin\phi$$

These are satisfied by solutions of the form

$$u_1 = k_1\, e^{mx} \qquad u_2 = K_2\, e^{mx}$$

which give

$$m^2 = \frac{A+C}{2(1-BD)}\left(1 \pm \sqrt{1 - \frac{4AC(1-BD)}{(A+C)^2}}\right)$$

When these solutions are compared with those obtained experimentally by Zangar, there is a discrepancy not quite as large as that given by equation (68).

13. Flexible Wall. The foregoing analysis may also be used to estimate the effect of wall flexibility on the water pressures. Suppose water is retained by a vertical cantilever wall which is sufficiently stiff so that wave propagation in the wall may be neglected. Using the same method of analysis and the notation as shown in Figure 27 the following equations apply for a sinusoidal vibration.

Horizontal displacement = $u \cdot f(y)\ \sin\omega t$

$$v = \frac{\partial u}{\partial x}\int_y^h f(y)\,dy \quad \sin\omega t$$

Applying Hamilton's Principle we obtain

$$\frac{d^2u}{dx^2} + \frac{A}{B} u = 0$$

$$u = u_0 \; e^{-\sqrt{\frac{A}{B}}\, x}$$

$$A = \int_0^h (f(y))^2 \, dy$$

$$B = \int_0^h \left(\int_y^h f(y) \, dy \right)^2 dy$$

The pressure on the wall is

$$p_w = \rho \dot{u}_0 \, \omega^2 \sqrt{\frac{A}{B}} \int_0^y \int_y^h f(y) \, dy \, dy \quad \sin \omega t$$

and the resultant force on the wall is

$$P = \rho \dot{u}_0 \, \omega^2 \sqrt{\frac{A}{B}} \int_0^h \int_0^y \int_y^h f(y) \, dy \, dy \, dy \quad \sin \omega t$$

For a wall of uniform cross-section, if we approximate the actual pressure by $p_0 \sin \frac{\pi}{2} \frac{y}{h}$ we obtain for $f(y)$:

$$f(y) = u_0 \left(1 - \beta + \beta \sin \frac{\pi}{2} \frac{x}{h} \right)$$

where

$$\beta = \frac{P}{u_0} \frac{h^3}{\left(\frac{\pi}{2}\right)^4 EI}$$

P = total force on wall.

The pressure and force are computed to be

$$p_w = \rho h \dot{u}_0 \omega^2 \sqrt{3} \sqrt{\frac{1-1.68\beta+1.18\beta^2}{1+2.44\beta+1.63\beta^2}} \left((1-\beta)\left(\frac{y}{h}-\frac{1}{2}\left(\frac{y}{h}\right)^2\right)+\left(\frac{2}{\pi}\right)^2 \beta \sin\frac{\pi}{2}\frac{y}{h}\right)$$

$$P = \frac{\rho h^2 \dot{u}_0 \omega^2}{\sqrt{3}} \sqrt{\frac{1-1.68\beta+1.18\beta^2}{1+2.44\beta+1.63\beta^2}} \left(1-0.22\beta\right)$$

This last equation may be written

$$\sqrt{3}\, K = \sqrt{\frac{1-1.68\beta+1.18\beta^2}{1+2.44\beta+1.63\beta^2}} \left(\frac{1-0.22\beta}{\beta}\right)$$

$$K = \left(\frac{\pi}{2}\right)^4 \frac{EI}{\rho \omega^2 h^5}$$

For a given wall and base acceleration, that is, a given K, this equation gives the appropriate value of β. Figure 28 gives a graph of K vs. β and also shows how the total force on the wall is reduced by wall flexibility.

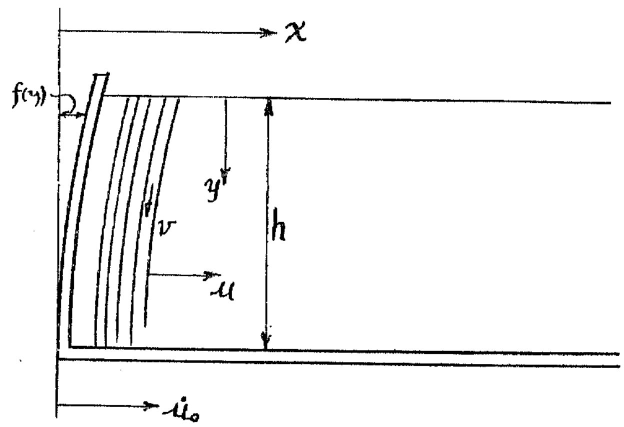

FIGURE 27

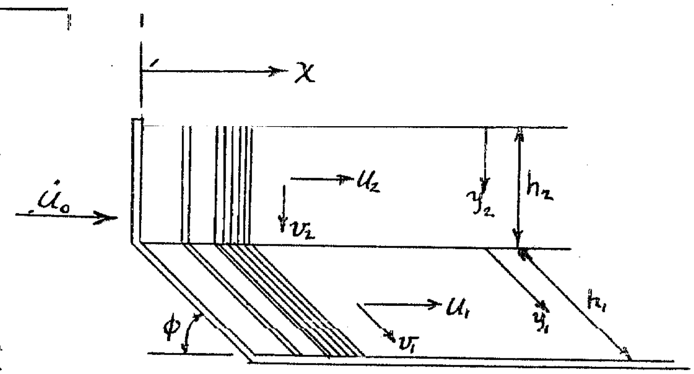

FIGURE 29

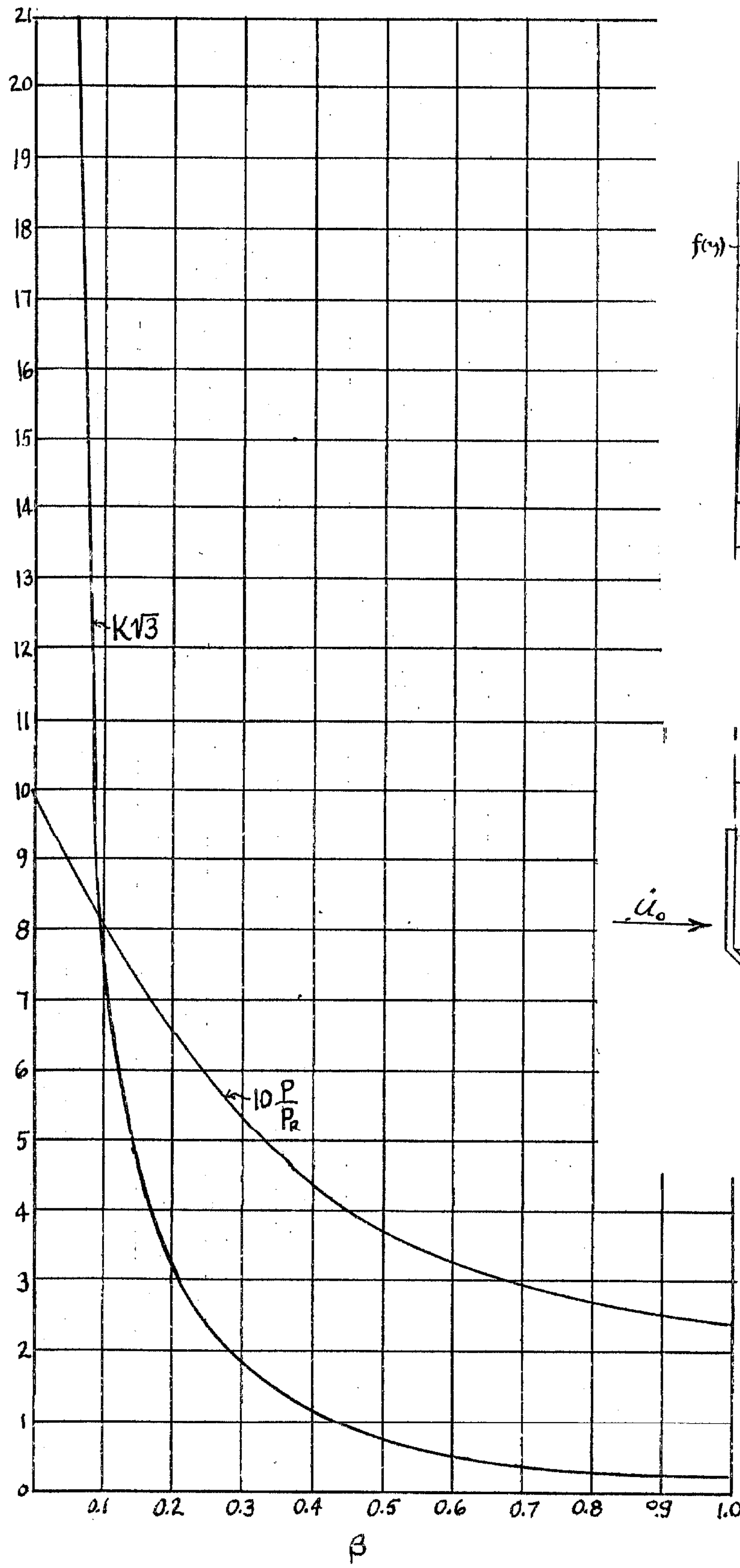

FIGURE 28

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